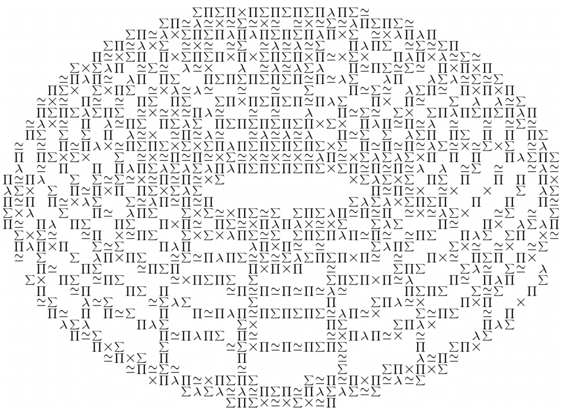
Cornell University, Math 6540

Simplicial categories

Instructors: Prof. Z



Varinderjit Mann

2022 Spring

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1 sSet as a sSet-enriched category

1.1 Preliminaries

- Symmetric monoidal categories.
- Basic definition of enrichment over monoidal categories.
- Theory of Anodyne extension in **sSet**. Refer Chapter-1 Section-4 in Jardine.
- Some technical details will be skipped in the interest of time (especially toward the end). And especially those that detract from the main topic at hand.
- We mainly use the language of Jardine - Simplicial Homotopy theory.
- Some references are also taken from Chapter-3 and Chapter-10 of Emily's Categorical Homotopy Theory book.
- Main duration of time will be given to discussing **sSet** as being a simplicial model category.

1.2 sSet as a symmetric monoidal category

We can use the categorical product of two simplices denoted by $\times$ as the tensor product functor $\otimes : \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{sSet}$. That is, $K \otimes L = K \times L$. This product is symmetric, associative and unital with the terminal object $*$.

Recall that **sSet** is a category of presheaves. Every small limit and colimit is computed element-wise. That is, $(K \times L)_n = K_n \times L_n$ with obvious face and degeneracy maps. This ensures the symmetric, associative and unital properties inherited from cartesian product of sets.

We do not go into the details of this verification.

1.3 sSet as sSet enriched category

There is a notion of internal hom in **sSet**, denoted by

$$[\cdot, \cdot] : \mathbf{sSet}^{op} \times \mathbf{sSet} \longrightarrow \mathbf{sSet}$$

and defined as

$$[K, L]_n = \hom_{\mathbf{sSet}}(\Delta^n \times K, Y)$$

The structure maps are defined by precomposition in the natural sense. Let $\theta : [m] \to [n]$. Then then corresponding structure map is given as

$$\hom_{\mathbf{sSet}}(\Delta^n \times K, Y) \to \hom_{\mathbf{sSet}}(\Delta^m \times K, Y)$$

$$(\Delta^n \times K \to Y) \mapsto (\Delta^m \times K \xrightarrow{\theta\times 1} \Delta^n \times K \to Y)$$

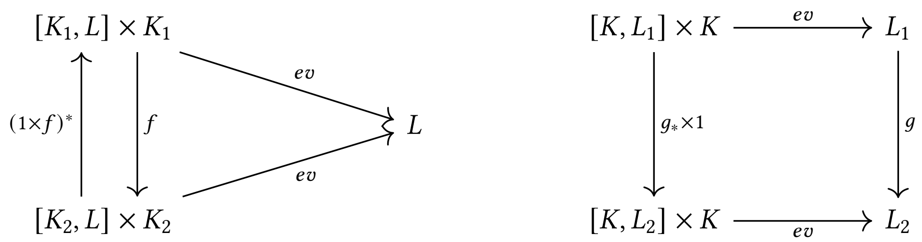
To see the functoriality of $[\cdot,\cdot]$. Pick any map $f : K_1 \to K_2$. This defines a map $[K_2, L] \to [K_1, L]$ by levelwise precomposition with $1 \times f$. Likewise, for any map $g : L_1 \to L_2$, we can define a map from $[K, L_1] \to [K, L_2]$ by post composition.

Part of the definition of internal hom also requires that we obtain an adjoint pair $(\cdot \times K) \dashv [K, \cdot]$ for all $K \in \mathbf{sSet}$. That is, we want to show the natural isomorphism

$$\hom_{\mathbf{sSet}}(X \times K, L) \cong \hom_{\mathbf{sSet}}(X, [K, L])$$

which is natural in X, L. (Also natural in K in addition).

This will require us to define another map called the evaluation map $ev : [K, L] \times K \to L$. The map is defined by sending any $(f, k) \mapsto f(i_n, k)$ where $i_n \in \Delta^n$ is the unique n-simplex. It is straightforward to check that ev is a simplicial map. Moreover, ev is also natural in both K and L in the following sense:



The first diagram says that $ev((1 \times f)^*\alpha_2, k_1) = ev(\alpha_2, f(k_1))$ and the second implies that $g \circ ev(\alpha_1, k_1) = ev((g_* \times 1)(\alpha_1, k_1))$

Finally, we define the map $ev_* : \hom_{\mathbf{sSet}}(X, [K, L]) \to \hom_{\mathbf{sSet}}(X \times K, L)$ as post composition with ev_* in the following sense:

$$(X \xrightarrow{g} [K, L]) \mapsto (X \times K \xrightarrow{g\times 1} [K, L] \times K \xrightarrow{ev} L)$$

This map has the following inverse. Pick any $h' : X \times K \to L$. Define the map $h : X \to [K, L]$ by setting

$$h(x) = \qquad \Delta^n \times K \xrightarrow{\alpha_x \times 1} X \times K \xrightarrow{h'} L$$

The evaluation map $ev : [K, \cdot] \times K \to \cdot$ defines the counit of the adjunction as it is the transpose of the identity map $[K, \cdot] \to [K, \cdot]$. This data with $\hom_{\mathbf{sSet}}$ isomorphism is sufficient to claim the adjunction. Moreover, the naturality in K follows from the (tedious) diagram

$$\begin{array}{ccc} \hom_{\mathbf{sSet}}(X \times K_1, L) & \xleftarrow{\;ev_*\;} & \hom_{\mathbf{sSet}}(X, [K_1, L]) \\ \uparrow{\scriptstyle (1\times f)^*} & & \uparrow{\scriptstyle ((1\times f)^*)_*} \\ \hom_{\mathbf{sSet}}(X \times K_2, L) & \xleftarrow{\;ev_*\;} & \hom_{\mathbf{sSet}}(X, [K_2, L]) \end{array}$$

We also find that the functor $[\cdot, L] : \mathbf{sSet}^{op} \to \mathbf{sSet}$ is **mutually right adjoint** to itself. Refer to Emily's book Chapter-4 Section-3.

That is, we have the following chain of natural isomorphisms:

$$\hom_{\mathbf{sSet}}(X, [K, L]) \cong \hom_{\mathbf{sSet}}(X \times K, L) \cong \hom_{\mathbf{sSet}}(K \times X, L) \cong \hom_{\mathbf{sSet}}(K, [X, L])$$

where the center natural isomorphism is induced by the natural isomorphism $\tau : X \times K \to K \times X$ by flipping. Recall that $X \times K$ level wise is just cartesian product of Sets.

By virtue of above, we would also like to label $[K, L] = L^K$, which is called the power object. As you might guess, hom object and power object need not always be same in the general case.

Lastly, we justify the notion of calling the functor $[\cdot, \cdot]$ as internal "hom". In some sense, it also satisfy an "adjunction".

$$[X \times K, L] \cong [X, [K, L]] \cong [K, [X, L]]$$

which is natural in X, K, L. The proof follows by noting that $[X \times K, L]$ and $[X, [K, L]]$ are representably isomorphic. The above is what we call as the property of being tensored and cotensored over **sSet**.

Consequentially, we define another notation for internal hom given by $\mathbf{hom}_{\mathbf{sSet}}(\cdot, \cdot) = [\cdot, \cdot]$. We also coin the term function complex for this.

Where does the enrichment come? Let $f : \Delta^n \times A \to B$ and $g : \Delta^n \times B \to C$. We can define a composition pairing as:

$$\circ : \mathbf{hom}_{\mathbf{sSet}}(B, C) \times \mathbf{hom}_{\mathbf{sSet}}(A, B) \to \mathbf{hom}_{\mathbf{sSet}}(A, C)$$

giving the map

$$\Delta^n \times A \xrightarrow{d \times 1} (\Delta^n \times \Delta^n) \times A \cong \Delta^n \times (\Delta^n \times A) \xrightarrow{1 \times f} \Delta^n \times B \xrightarrow{g} C$$

where $d : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal map.

It can be verified that the pairing is associative and unital with the image of $* \to [B, B]$ where $*$ is the terminal object in **sSet** with the map defined by sending the vertex $* \mapsto 1_B$. In other words, we are picking the simplicial subset generated by the vertex 1_B.

All of the above information can be condensed by claiming that **sSet** is <u>enriched over</u> **sSet** with pairing $\circ$ that is associative and unital with the unit object described above. But a little more is true, **sSet** is also tensored and cotensored over itself such that the underlying category is **sSet**.

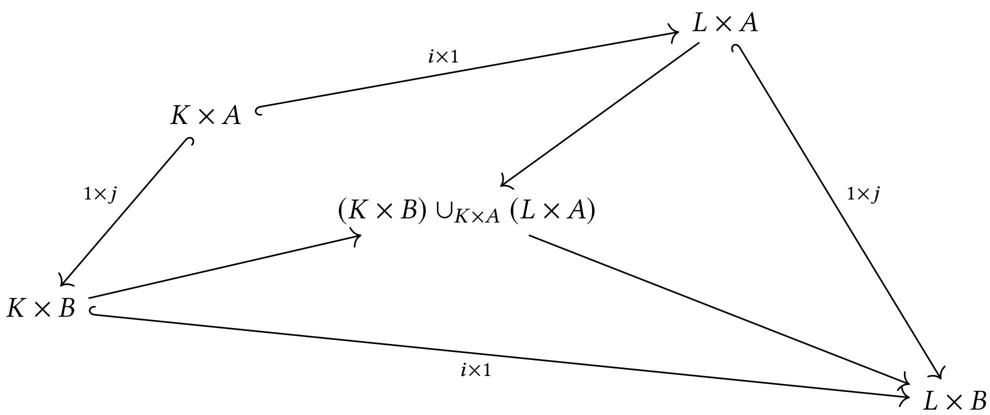
The reason why function complexes are so important is their relation to the theory of anodyne maps in **sSet**. From now on, we use the standard Quillen model structure on simplicial sets. We note the following result first without a complete proof.

Theorem 1.1. *Let $i : K \to L$ be a cofibration and $j : A \to B$ be a cofibration for some $K, L, A, B \in$ **sSet**. Then the map*

$$(i \times 1) \cup (1 \times j) : (K \times B) \cup_{(K \times A)} (L \times A) \to L \times B$$

is a cofibration which is trivial if either j or i is.

Proof. The map is induced by the pushout square



It can be checked that this map is a cofibration by working levelwise.

The part regarding j or i being trivial is a bit more complicated. We need a dual notion to the above diagram. To this end, we obtain function complexes naturally. ◻

Define the axiom **SM7** as the following:

Suppose $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration. Then the map

$$\mathbf{hom}_{\mathbf{sSet}}(B, X) \xrightarrow{(j^*, q_*)} \mathbf{hom}_{\mathbf{sSet}}(A, X) \times_{\mathbf{hom}_{\mathbf{sSet}}(A,Y)} \mathbf{hom}_{\mathbf{sSet}}(B, Y)$$

is a fibration. Moreover, it is trivial if j or q is.

Lemma 1.2. *Axiom **SM7** is equivalent to Theorem 1.1*

Proof. Pick any cofibration $i : K \to L$ and consider the commuting square:

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & \mathbf{hom}_{\mathbf{sSet}}(B,X) \arrow[d, "(j^*,q_*)"] \\ L \arrow[r] & \mathbf{hom}_{\mathbf{sSet}}(A,X) \times_{\mathbf{hom}_{\mathbf{sSet}}(A,Y)} \mathbf{hom}_{\mathbf{sSet}}(B,Y) \end{tikzcd}$$

This commuting square is equivalent to writing

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & \mathbf{hom}_{\mathbf{sSet}}(B,X) \arrow[r, "j^*"] \arrow[d, "q_*"] & \mathbf{hom}_{\mathbf{sSet}}(A,X) \arrow[d, "q_*"] \\ L \arrow[r] \arrow[urr] & \mathbf{hom}_{\mathbf{sSet}}(B,Y) \arrow[r, "j^*"] & \mathbf{hom}_{\mathbf{sSet}}(A,Y) \end{tikzcd}$$

Using the internal hom and product adjunction (which was natural in all three variables), we can write the dual diagram

$$\begin{tikzcd} K \times A \arrow[r, "1\times j"] \arrow[d, "i\times 1"'] & K \times B \arrow[r] \arrow[d, "i\times 1"] & X \arrow[d, "q"] \\ L \times A \arrow[r, "1\times j"'] \arrow[urr] & L \times B \arrow[r] & Y \end{tikzcd}$$

Finally, the above diagram is equivalent to the following:

$$\begin{tikzcd} (K \times B) \cup_{(K\times A)} (L \times A) \arrow[r] \arrow[d, "(i\times 1)\cup(1\times j)"'] & X \arrow[d, "q"] \\ L \times B \arrow[r] & Y \end{tikzcd}$$

Now we just use the lifting properties of fibrations and cofibrations.

□

Theorem 1.3. *A simplicial map $g : W \longrightarrow Z$ is a trivial fibration iff it has the right lifting property with respect to all boundary inclusions $\partial\Delta^n \subseteq \Delta^n$ for $n \geq 0$.*

Theorem 1.4. *A simplicial map $g : W \longrightarrow Z$ is a fibration iff it has the right lifting property with respect to all horn inclusions $\Lambda^n_k \subseteq \Delta^n$ for all $n \geq 0$.*

Theorem 1.4 might be regarded as a definition depending upon your perspective.

This result will be skipped as theorem 1.3 requires the theory of minimal fibrations. But it is central to the proof of Theorem 1.1. Instead of proving theorem 1.1, we prove the **SM7** axiom as it is much simpler in light of Theorem 1.3 and 1.4.

Proof. **Axiom SM7**:

Consider any diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(B, X) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle (j^*, q_*)} \\ \Delta^n & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(A, X) \times_{\mathbf{hom}_{\mathbf{sSet}}(A,Y)} \mathbf{hom}_{\mathbf{sSet}}(B, Y) \end{array}$$

We want to show that this diagram has a lift. This is equivalent to asking that the following diagram must have a lift

$$\begin{array}{ccc} (\Lambda^n_k \times B) \cup_{(\Lambda^n_k \times A)} (\Delta^n \times A) & \longrightarrow & X \\ {\scriptstyle (i\times 1)\cup(1\times j)}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \Delta^n \times B & \longrightarrow & Y \end{array}$$

However, it is known that the map on left is an anodyne map because $\Lambda^n_k \subseteq \Delta^n$ is anodyne and $A \to B$ is a cofibration. Consequentially, the lift exists and the desired map is always a fibration.

Now suppose that q is trivial and consider any diagram whose lift we desire

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(B, X) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle (j^*, q_*)} \\ \Delta^n & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(A, X) \times_{\mathbf{hom}_{\mathbf{sSet}}(A,Y)} \mathbf{hom}_{\mathbf{sSet}}(B, Y) \end{array}$$

This is equivalent to the diagram

$$\begin{array}{ccc} (\partial\Delta^n \times B) \cup_{(\partial\Delta^n \times A)} (\Delta^n \times A) & \longrightarrow & X \\ {\scriptstyle (i\times 1)\cup(1\times j)}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \Delta^n \times B & \longrightarrow & Y \end{array}$$

We have already shown that the map on the left is a cofibration. The map on the right is a trivial fibration by assumption and so the lift exists.

Finally suppose that j is trivial and consider any diagram whose lift we desire

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(B,X) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle (j^*,q_*)} \\ \Delta^n & \longrightarrow & \mathbf{hom}_{\mathbf{sSet}}(A,X) \times_{\mathbf{hom}_{\mathbf{sSet}}(A,Y)} \mathbf{hom}_{\mathbf{sSet}}(B,Y) \end{array}$$

This is equivalent to demanding a lift for

$$\begin{array}{ccc} (\partial\Delta^n \times B) \cup_{(\partial\Delta^n \times A)} (\Delta^n \times A) & \longrightarrow & X \\ {\scriptstyle (i\times 1)\cup(1\times j)}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \Delta^n \times B & \longrightarrow & Y \end{array}$$

Since j is anodyne, the map on the left is also anodyne. Therefore, the desired lift exists.

□

Consequentially, Theorem 1.1 is proven as desired. The key takeaway is that the function complexes somehow encode a special relation with the anodyne maps of simplicial sets. Theorem 1.1 states a property which we would take intuitively when dealing with CW-complexes, and which is justified below. In some sense, it is a desirable property that we would like to have for any "simplicially enriched" category.

There is an equivalent axiom **SM7b** which is easier to verify. The 'b' refers to it being dual.

Let $j : A \to B$ be a cofibration for some $A, B \in \mathbf{sSet}$. Then the map

$$(i \times 1) \cup (1 \times j) : (\partial\Delta^n \times B) \cup_{(\partial\Delta^n \times A)} (\Delta^n \times A) \to \Delta^n \times B$$

is a cofibration, which is trivial if j is. And the map

$$(i \times 1) \cup (1 \times j) : (\Lambda^n_k \times B) \cup_{(\Lambda^n_k \times A)} (\Delta^n \times A) \to \Delta^n \times B$$

is a trivial cofibration.

Proof. Use Theorem 1.3 and 1.4 □

With the axiom **SM7** and the simplicial enrichment, we form **sSet** as a <u>simplicial model category</u>. That is, **sSet** is a simplicially enriched category whose model structure interact with the simplicial enrichment by the axiom **SM7**.

2 Simplicial Model Categories

2.1 sSet enriched categories

Keeping in mind the desired axiom **SM7b**, we want to provide any category C such a structure.

We call a category C as **sSet**-enriched if it has a "Mapping space" functor:

$$\mathbf{hom}_C(\cdot,\cdot) : C^{op} \times C \to \mathbf{sSet}$$

such that for all $A, B \in C$ and $K, L \in \mathbf{sSet}$, we have the following three conditions:

- $\hom_{\mathbf{sSet}}(*, \mathbf{hom}_C(A,B)) \cong \mathbf{hom}_C(A,B)_0 = \hom_C(A,B)$. Which can also be translated as saying that the underlying category is C itself.
- The functor $\mathbf{hom}_C(A,\cdot) : C \to \mathbf{sSet}$ has a left adjoint $\cdot \otimes A : \mathbf{sSet} \to C$ such that there is a natural isomorphism $(K \times L) \otimes A \cong K \otimes (L \otimes A)$ which is natural in all three variables. In other words, $\hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A,B)) \cong \hom_C(K \otimes A, B)$ which is natural in K and B.

  Moreover, $K \otimes \cdot : C \to C$ defines a functor by sending the map $f : A_1 \to A_2$ to a map $1 \otimes f : K \otimes A_1 \to K \otimes A_2$ so that it correspond to the map $\hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A_2, B)) \to \hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A_1, B))$ induced by f via the natural isomorphism. In particular, we extended the isomorphism to be natural in all three variables.
- The functor $\mathbf{hom}_C(\cdot, B) : C^{op} \to \mathbf{sSet}$ has a mutual right adjoint $B^{-} : \mathbf{sSet}^{op} \to C$. In particular, we have a natural isomorphism $\hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A,B)) \cong \hom_C(A, B^K)$ which is natural in K and A.

  Moreover, $\cdot^K : C \to C$ defines a functor by sending the map $f : B_1 \to B_2$ to a map $f^K : B_1^K \to B_2^K$ so that it correspond to the map $\hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A, B_1)) \to \hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A, B_2))$ induced by f via the natural isomorphism. In particular, we extended the isomorphism to be natural in all three variables.

Precisely speaking we should be calling the category C as closed **sSet**-module. It is also known that each closed **sSet**-module correspond to a unique **sSet**-enriched category that is tensored and cotensored (we will see this later), and whose underlying adjunction of the **sSet**-enriched tensor-cotensor-hom adjunction is the given adjunction unenriched (**Set**) adjunction. Refer to chapter 10 Section 1 in Emily's Categorical Homotopy Theory and also chapter 3 section 7.

We begin by noting some immediate consequences of the above definition.

Lemma 2.1. *There is an adjoint pair* $(K \otimes \cdot) \dashv \cdot^K$.

Proof. This follows from the chain of natural isomorphisms

$$\hom_C(K \otimes A, B) \cong \hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A,B)) \cong \hom_C(A, B^K)$$

which is natural in all three variables. □

Lemma 2.2. *There is a natural isomorphism $B^{L\times K} \cong B^{K\times L} \cong (B^K)^L \cong (B^L)^K$ for all $K, L \in \mathbf{sSet}$ and $B \in C$.*

Proof. The center one follows from the following chain of natural ismorphisms for any $A \in C$.

$$\hom_C(A, B^{K\times L}) \cong \hom_C((K \times L) \otimes A, B)$$
$$\cong \hom_C(K \otimes (L \otimes A), B) \cong \hom_C(L \otimes A, B^K) \cong \hom_C(A, (B^K)^L)$$

The rest follows from using the natural isomorphism $K \times L \cong L \times K$. □

Lemma 2.3. *For all $n \geq 0$, we have $\mathbf{hom}_C(A,B)_n \cong \hom_C(\Delta^n \otimes A, B)$*

Proof. By Yoneda Lemma, we have

$$\mathbf{hom}_C(A,B)_n \cong \hom_{\mathbf{sSet}}(\Delta^n, \mathbf{hom}_C(A,B)) \cong \hom_C(\Delta^n \otimes A, B)$$

□

Our first goal is to discuss why this definition is equivalent to claiming that there is an enrichment of C over **sSet** in the sense that there is a associative pairing

$$\circ : \mathbf{hom}_C(B,C) \times \mathbf{hom}_C(A,B) \to \mathbf{hom}_C(A,C)$$

with a natural choice of unit.

Let $f \in \mathbf{hom}_C(A,B)_n$ and $g \in \mathbf{hom}_C(B,C)_n$. We can define a composition pairing as giving the map

$$\Delta^n \otimes A \xrightarrow{d\times 1} (\Delta^n \times \Delta^n) \otimes A \cong \Delta^n \otimes (\Delta^n \otimes A) \xrightarrow{1\times f} \Delta^n \otimes B \xrightarrow{g} C$$

where $d : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal map. This pair is associative and unital with the unit being the image of $* \to \mathbf{hom}_C(A,A)$ where $*$ is the terminal object in **sSet** with the map defined by sending the vertex $* \mapsto 1_A$.

In particular, C is enriched over **sSet** in the expected sense.

There is also an analogue of internal hom adjunction as in the case of simplicial sets. And this is what guarantees the property of being tensored and cotensored over **sSet**.

Lemma 2.4. *We have the following natural isomorphisms (natural in $K \in \mathbf{sSet}$ and $A, B \in C$)*

- $\mathbf{hom}_{\mathbf{sSet}}(K, \mathbf{hom}_C(A, B)) \cong \mathbf{hom}_C(K \otimes A, B)$
- $\mathbf{hom}_{\mathbf{sSet}}(K, \mathbf{hom}_C(A, B)) \cong \mathbf{hom}_C(A, B^K)$

Proof. Write $\mathbf{hom}_{\mathbf{sSet}}(\cdot, \cdot)$ as $[\cdot, \cdot]$ like in our older notation.

Choose any $L \in \mathbf{sSet}$ and note the following:

$$\hom_{\mathbf{sSet}}(L, [K, \mathbf{hom}_C(A, B)]) \cong \hom_{\mathbf{sSet}}(L \times K, \mathbf{hom}_C(A, B)) \cong \hom_{\mathbf{sSet}}((L \times K) \otimes A, B)$$

$$\cong \hom_{\mathbf{sSet}}(L \otimes (K \otimes A), B) \cong \hom_{\mathbf{sSet}}(L, \mathbf{hom}_C(K \otimes A, B))$$

Therefore, we are done. □

There is an equivalent important formulation (only for **sSet**) in terms of the existence of the functors $\cdot \otimes \cdot : \mathbf{sSet} \times C \to C$ and $\cdot^{-} : \mathbf{sSet}^{op} \times C \to C$

Lemma 2.5. *Let C be a category equipped with a functor*

$$\cdot \otimes \cdot : \mathbf{sSet} \times C \to C$$

such that the following holds:

1. *For a fixed $K \in \mathbf{sSet}$, $K \otimes \cdot : C \to C$ has a right adjoint $\cdot^K : C \to C$.*
2. *For fixed $A \in C$, the functor $\cdot \otimes A : \mathbf{sSet} \to C$ commutes with arbitrary colimits and $A \otimes * \cong A$.*
3. *$(K \times L) \otimes A \cong K \otimes (L \otimes A)$ naturally in $K, L \in \mathbf{sSet}$ and $A \in C$.*

Then C is a simplicially enriched category with

$$\mathbf{hom}_C(A, B)_n = \hom_C(\Delta^n \otimes A, B)$$

Proof. By the density theorem $K = \operatorname{colim} \Delta^n$ (over the category of simplices of K) we have,

$$\hom_{\mathbf{sSet}}(K, \mathbf{hom}_C(A, B)) \cong \lim \hom_{\mathbf{sSet}}(\Delta^n, \mathbf{hom}_C(A, B)) \cong \lim \mathbf{hom}_C(A, B)_n$$

$$= \lim \hom_C(\Delta^n \otimes A, B) \cong \hom_C(\operatorname{colim}(\Delta^n \otimes A), B) \cong \hom_C(K \otimes A, B)$$

This provides the desired natural isomorpism. Rest is straightforward by using the given adjunction in (1), that is, $\hom_C(K \otimes A, B) \cong \hom_C(A, B^K)$. □

We now return back to the example to **sSet** to makes thing clear.

Lemma 2.6. **sSet** *is simplicially enriched by the two functors*

$$K \otimes A = K \times A$$

$$B^K = [K, B]$$

2.2 Simplicial model categories

We can now discuss the notion of generalized axiom **SM7**.

Definition 1. *Axiom **SM7**: Let C be a model category which is* **sSet** *enriched. For $A, B, X, Y \in C$, let $j : A \longrightarrow B$ be a cofibration and $q : X \longrightarrow Y$ a fibration. Then,*

$$\mathbf{hom}_C(B, X) \xrightarrow{(j^*, q_*)} \mathbf{hom}_C(A, X) \times_{\mathbf{hom}_C(A,Y)} \mathbf{hom}_C(B, Y)$$

is a fibration. Moreover, it is trivial if j or q is trivial.

Such categories satisfying **SM7** will be called simplicial model categories.

Just as in the case of simplicial sets, we have the following theorem:

Theorem 2.7. *Axiom **SM7** is equivalent to the following:*

Let $i : K \longrightarrow L$ be a cofibration in **sSet** *and $j : A \longrightarrow B$ be a cofibration in C. Then the map*

$$(i \otimes 1) \cup (1 \otimes j) : (K \otimes B) \cup_{(K \otimes A)} (L \otimes A) \longrightarrow L \otimes B$$

is a cofibration. Moreover, it is trivial if either j or i is trivial.

However, in the general case, there is one more equivalent notion involving the functor B^K.

Theorem 2.8. *Axiom **SM7** is equivalent to the following:*

Let $i : K \longrightarrow L$ be a cofibration in **sSet** *and $q : X \longrightarrow Y$ a fibration in C. Then,*

$$X^L \xrightarrow{(X^i, q^L)} X^K \times_{Y^K} Y^L$$

is a fibration. Moreover, it is trivial if either i or q is trivial.

There are also similar weakening of the requirement as done in the case of simplicial sets.

Lemma 2.9. *Axiom **SM7** is equivalent to the following Axiom **SM7b**:*

Let $j : A \longrightarrow B$ be a cofibration in C. Then the map

$$(i \otimes 1) \cup (1 \otimes j) : (\partial\Delta^n \otimes B) \cup_{(\partial\Delta^n \otimes A)} (\Delta^n \otimes A) \longrightarrow \Delta^n \otimes B$$

is a cofibration, which is trivial if j is. And the map

$$(i \otimes 1) \cup (1 \otimes j) : (\Lambda^n_k \otimes B) \cup_{(\Lambda^n_k \otimes A)} (\Delta^n \otimes A) \longrightarrow \Delta^n \otimes B$$

is a trivial cofibration.

Lemma 2.10. *Axiom **SM7** is equivalent to the following Axiom **SM7a**:*

Let $q : X \to Y$ a fibration in C. Then the map

$$X^{\Delta^n} \xrightarrow{(X^i, q^L)} X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

is a fibration, which is trivial if j is. And the map

$$X^{\Delta^n} \xrightarrow{(X^i, q^L)} X^{\Lambda^n_k} \times_{Y^{\Lambda^n_k}} Y^{\Delta^n}$$

is a trivial fibration.

An important consequence homotopy theoretic consequence of **SM7** is the existence of a natural choice of good cylinder object $\Delta^1 \otimes A$ for cofibrant $A \in C$ and good path object X^{Δ^1} for fibrant $X \in C$. We work toward this by noting the following lemmas.

Lemma 2.11. *Let C be a simplicial model category and $q : X \to Y$ be a fibration. Let $\phi \to B$ be a cofibration (or B is cofibrant). Then,*

$$q_* : \mathbf{hom}_C(B, X) \to \mathbf{hom}_C(B, Y)$$

is a fibration. Which is trivial if q is.

Similarly, if $j : A \to B$ is a cofibration and $X \to *$ is a fibration (or X is fibrant). Then,

$$j^* : \mathbf{hom}_C(B, X) \to \mathbf{hom}_C(A, X)$$

is a fibration. Which is trivial if j is.

As usual, we can use the adjoint relation to form equivalent notion in terms of B^K and $K \otimes A$ functors.

Lemma 2.12. *Let C be a simplicial model category. Let $\phi \to B$ be a cofibration in C (or B is cofibrant) and $i : K \to L$ be a cofibration in* **sSet**. *Then,*

$$i \otimes 1 : K \otimes B \to L \otimes B$$

is a cofibration in C. Which is trivial if i is.

Similarly, if $X \to *$ is a fibration (or X is fibrant). Then,

$$X^i : X^L \to X^K$$

is a fibration in C. Which is trivial if i is.

Define the map $d^1 \sqcup d^0 : B \sqcup B \to \Delta^1 \otimes B$ as

$$B \sqcup B \cong (* \otimes B) \sqcup (* \otimes B) \cong (* \sqcup *) \otimes B \cong \partial\Delta^1 \otimes B \xrightarrow{(d^1 \sqcup d^0)\otimes 1} \Delta^1 \otimes B$$

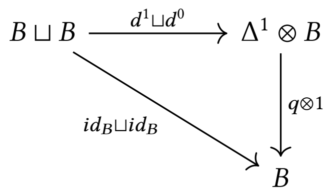
Since $\partial\Delta^1 \hookrightarrow \Delta^1$ is a cofibration, the map $d^1 \sqcup d^0 : B \sqcup B \to \Delta^1 \otimes B$ is a cofibration.

Also consider the map $q \otimes 1 : \Delta^1 \otimes B \to B$ defined using $q : \Delta^1 \to *$ as follow

$$q \otimes 1 : \Delta^1 \otimes B \to * \otimes B \cong B$$

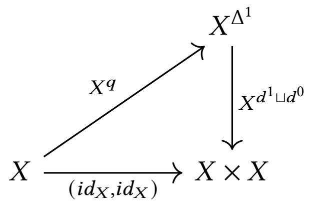
Then it is clear that $B \hookrightarrow B \sqcup B \xrightarrow{d^1 \sqcup d^0} \Delta^1 \otimes B \xrightarrow{q \otimes 1} B$ is identity. We know that $* \sqcup B \to \Delta^1 \sqcup B$ is a acyclic cofibration from lemma 1.16. Consequentially, by 2-of-3, $q \otimes 1$ is also a weak equivalence.

Lastly, we have the diagram



showing that we have got a good cylinder object.

On the other hand, we can also show that $X^* \cong X$ and $X^{\partial\Delta^1} \cong X \times X$. Consequentially, with similar proof as above, we will find that X^{Δ^1} is a good path ojbect.



2.3 Examples of sSet categories

There is a notion of simplicial objects just as there is a notion of simplicial sets. Let C be any category and Δ be the simplex category. We define $sC = [\Delta^{op}, C]$ as the functor category. Just as **sSet** could be naturally turned into a simplicially enriched category, one might expect sC to be turned as well. The only requirement is that we want C to be complete and co-complete (just like **Set** is). The assumption that C is co-complete allows us to define "coproduct" copower.

Let $K \in$ **sSet** and $X \in sC$. Then we define,

$$(K \otimes X)_n = \bigsqcup_{K_n} X_n$$

The structure maps are provided by $\phi : [m] \to [n]$ inducing the map:

$$\bigsqcup_{K_m} X_m \xrightarrow{\sqcup \phi^*} \bigsqcup_{K_m} X_n \longrightarrow \bigsqcup_{K_n} X_n$$

To form a functor $\cdot \otimes \cdot$, let $X_1 \to X_2$ be a map in sC. This defines the levelwise map:

$$\bigsqcup_{K_n} (X_1)_n \longrightarrow \bigsqcup_{K_n} (X_2)_n$$

Let $K_1 \to K_2$ be a map in **sSet**. This defines a levelwise map:

$$\bigsqcup_{(K_1)_n} (X)_n \longrightarrow \bigsqcup_{(K_2)_n} (X)_n$$

where appropriate $(K_1)_n$ copy of X_n is mapped to an appropriate $(K_2)_n$ copy of X_n according to the simplicial map $K_1 \to K_2$.

Theorem 2.13. *Let C be both complete and cocomplete. The coproduct tensor functor $\cdot \otimes \cdot : \mathbf{sSet} \times sC \to sC$ allows us to define sC as a simplicially enriched category with*

$$\mathbf{hom}_{sC}(X, Y)_n = \hom_{sC}(\Delta^n \otimes X, Y)$$

Proof. Quite long and technical so we skip it. Refer to Jardine Chapter-2 Theorem 2.5. □

Theorem 1.17 allows to define various examples. Pick C to be any of **Grp**, **AbGrp**, **Rings**, $\mathbf{R-mod}$, etc. Then sC is a simplicially enriched category.

2.4 Examples of simplicial model categories

Just as before, we'd like to discuss when we can form sC into not only a simplicially enriched category but also a simplicial model category. There are many ways to form one. However, if we wish to use the simplicial enrichment structure described in theorem 1.17, we'd like to discuss when the simplicially enriched structure is preserved by an adjunction in a certain sense. Let $C : F \rightleftarrows G : \mathcal{D}$ be an adjunction with $F \dashv G$ between two simplicially enriched categories C and $\mathcal{D}$.

Lemma 2.14. *Suppose that for all $K \in$ **sSet** and $A \in C$, there is a natural isomorphism $F(K \otimes A) = K \otimes F(A)$. Then,*

1. *The adjunction extends to a natural isomorphism*

$$\mathbf{hom}_{\mathcal{D}}(FA, B) \cong \mathbf{hom}_{C}(A, GB)$$

2. *For all $K \in$ **sSet** and $B \in \mathcal{D}$, there is a natural isomorphism*

$$G(B^K) \cong G(B)^K$$

Proof. 1. Note that

$$\mathbf{hom}_{\mathcal{D}}(FA, B)_n \cong \hom_{\mathcal{D}}(\Delta^n \otimes FA, B) \cong \hom_{\mathcal{D}}(F(\Delta^n \otimes A), B)$$

2. Note that

$$\hom_{C}(A, G(B^K)) \cong \hom_{C}(FA, B^K) \cong \hom_{C}(K \otimes FA, B) \cong \hom_{C}(F(K \otimes A), B)$$

□

Consider any category sC such that there is a pair of adjoints $G : sC \rightleftarrows \mathbf{sSet} : F$ with $F \dashv G$. Define the "tentative" model structure on sC by setting any morphism $f : A \to B$ in sC to be:

1. a weak equivalence if Gf is a weak equivalence in **sSet**
2. a fibration if Gf is a fibration in **sSet**
3. a cofibration if it has the left lifting property with respect to all trivial fibrations in sC.

Theorem 2.15. *If C is complete, co-complete, and G preserves filtered colimits then sC is a model category provided certain nice condition on cofibrations.*

Proof. Long and technical so we skip it. □

Theorem 2.16. *Let sC be as above in theorem* 1.19*. Then it is a simplicial model category with the simplicial enrichement provided by the theorem* 1.17*.*

Proof. Let $K, L \in \mathbf{sSet}$. We notice that $F(K \times L) = K \otimes F(L)$. This is because any cartesian product $K_n \times L_n$ of sets can be regarded as a coproduct of K_n copies of L_n and that F as a left adjoint preserves colimits. Consequentially, we will obtain that $G(B^K) \cong G(B)^K$ where $B^K \in sC$ and $G(B)^K \in \mathbf{sSet}$.

Pick any $X, Y \in sC$ such that $q : X \to Y$ is a fibration in sC. And let $i : K \to L$ be a cofibration in **sSet**. We want to show that the map

$$X^L \xrightarrow{(X^i, q^L)} X^K \times_{Y^K} Y^L$$

is a fibration in sC. And that, it is trivial in sC if i or q is trivial. From the definition of fibrations, weak equivalences in sC and that G as a right adjoint preserves limits, this is the same as checking that the map

$$G(X)^L \xrightarrow{(G(X)^i, G(q)^L)} G(X)^K \times_{G(Y)^K} G(Y)^L$$

is a fibration in **sSet**. And that, it is trivial in **sSet** if i or $G(q) : G(X) \to G(Y)$ is trivial. Recall that $G(q) : G(X) \to G(Y)$ is a (trivial)-fibration in **sSet** iff $q : X \to Y$ is a (trivial)-fibration sC by definition. Now the statement holds due to **SM7** of simplicial sets.

Theorem 1.20 allows us to consider many examples such as simplicial groups, simplicial abelian groups or simplicial R-modules by using G as the forgetful functor. Although, we do not go into the detail due to time.

□

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